INTERNATIONAL MATHEMATICS
TOURNAMENT OF TOWNS

SENIOR PAPER: YEARS 11,12

Tournament 41, Northern Autumn 2019 (O Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. An illusionist lays a full deck of 52 cards in a row and tells spectators that 51 cards will be taken away step by step with only the Three of Clubs remaining on the table. On each step some spectator tells the illusionist a number so that a card lying on the place with this number in the row is taken away. However, the illusionist makes his own decision from which side of the row, left or right, he should count that number from to take the card away. For which initial positions of the Three of Clubs can the illusionist guarantee the success of his trick for sure? (3 points)

2. Let $ABCDE$ be a convex pentagon such that AE is parallel to CD and $AB = BC$. Let the angle bisectors of angles A and C intersect at point K. Prove that BK is parallel to AE. (4 points)

3. An integer x written on a blackboard can be replaced either with $3x + 1$ or with $\lfloor x/2 \rfloor$ (the greatest integer not exceeding $x/2$). Prove that if 1 is initially written, then any positive integer can be obtained by using the operations above. (4 points)

4. In a polygon, any two adjacent sides are perpendicular. Two vertices of the polygon are called *unfriendly*, if the angle bisectors emanating from those vertices are perpendicular. Prove that for each vertex the number of vertices unfriendly with that vertex is even. (5 points)

5. There is a row of 100 squares each containing a counter. Any 2 neighbouring counters can be swapped for 1 dollar and any 2 counters that have exactly 4 counters between them can be swapped for free. What is the least amount of money that must be spent to rearrange the counters in reverse order? (5 points)